Elements of Signal Processing

Intro. Biomedical Imaging and Image Analysis

September 8, 2008
Motivation:

Although biomedical signals/images are in continuous time, much of modern processing performed digitally.
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Although biomedical signals/images are in continuous time, much of modern processing performed digitally.

Topics in this presentation:

- Sampling
- Reconstruction
- Applications
Sampling

**Figure:** From S. Mallat, *A wavelet Tour of Signal Processing*
Diracs

Definition
Let smooth function $\Psi(t)$ be such that $\int_{-\infty}^{\infty} \Psi(t) = 1$.

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Psi(t/\epsilon)$$
Diracs

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Example

$$\Psi_\epsilon(t) = \frac{1}{\epsilon \sqrt{\pi}} e^{-t^2/\epsilon^2}$$
Diracs

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Example

$$\Psi_{\epsilon}(t) = \frac{1}{\epsilon \sqrt{\pi}} e^{-t^2/\epsilon^2}$$

It associates any continuous function $f(t)$ its value at $t = 0$.

$$\lim_{\epsilon \to 0} \int \Psi_{\epsilon}(t) f(t) dt = f(0)$$
Symbolic Calculations

Translation

- \( \delta(t - u) \) is delta function \( \delta(t) \) translated by \( u \).
- Fourier Transform: \( \delta(u - t) \leftrightarrow e^{-j\omega u} \)
- Alternatively: \( e^{jwt} \leftrightarrow 2\pi \delta(\omega - v) \)
Symbolic Calculations

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Association

Associate $f(t)\delta(t)$ with $f(0)$ and $f(t)\delta(t - u)$ with $f(u)$ through definition above.
Symbolic Calculations

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Convolution

$\int f(t)\delta(u - t) = f(u)$ or $f \ast \delta(t) = f(t)$
Symbolic Calculations

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Convolution

$$\int f(t)\delta(u - t) = f(u) \text{ or } f * \delta(t) = f(t)$$

Sampling

$$f_T(t) = f(t)\sum_{n=-\infty}^{\infty} \delta(t - nT)$$
Poisson Formula

Let $f(t)$ be a function of sufficient smoothness and decay. Then

$$f_0(t) = \sum_{n=-\infty}^{\infty} f(t - nT)$$

is periodic with period $T$ and Fourier Series

$$f_0(t) = \sum_{k=-\infty}^{\infty} \left[ \frac{1}{T} \int_{-T/2}^{T/2} f_0(\tau) e^{-j2\pi k \tau / T} d\tau \right] e^{j2\pi kt / T}$$
Poisson Formula

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\]

Looking closer at the coefficients:

\[
    \int_{-T/2}^{T/2} f_0(\tau) e^{-j2\pi k\tau/T} d\tau = \sum_{n=-\infty}^{\infty} \int_{(2n-1)T/2}^{(2n+1)T/2} f(\tau) e^{-j2\pi k\tau/T} d\tau
\]
Poisson Formula

But

\[
\sum_{n=-\infty}^{\infty} \int_{(2n-1)T/2}^{(2n+1)T/2} f(\tau) e^{-j2\pi k\tau/T} d\tau = \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi k\tau/T} d\tau = \hat{f} \left( \frac{2\pi k}{T} \right)
\]

where \( \hat{f}(\omega) \) is the continuous time Fourier Transform of \( f(t) \).
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Poisson Formula

\[
f_0(t) = \sum_{n=-\infty}^{\infty} f(t - nT) = \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{2\pi k}{T}\right) e^{j2\pi kT/T}
\]
Let $f_T(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT)$. Then its continuous time Fourier transform is

$$
\hat{f}_T(\omega) = \sum_{n=-\infty}^{\infty} f(nT)e^{-jnT\omega}
$$

$$
= \frac{1}{T} \sum_{\omega = -\infty}^{\infty} \hat{f}(\omega - 2\pi k/T)
$$
Sampling

With $f_T(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT)$:

$$\hat{f}_T(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}(\omega - 2\pi k/T)$$

Figure: From S. Mallat, A Wavelet Tour of Signal Processing
Aliasing

Suppose the support of $\hat{f}(\omega)$ goes beyond $[-\pi/T, \pi/T]$. Then

![Diagram](image)

**Figure:** From S. Mallat, A Wavelet Tour of Signal Processing
Nyquist criterion

Let a signal $f(t)$ be band limited to $B$ (frequency in cycles per second). Then $\frac{\pi}{T(2\pi)} = \frac{1}{2T} = B$, or $\frac{1}{T} = 2B$ is the minimum frequency (in cycles per second) to avoid aliasing.
Aliasing

Example

![Aliasing Example Graph](image-url)
Aliasing

Example

Aliased

Figure: From Thevenaz et al., Interpolation and Resampling, Handbook of Medical Imaging. SPIE Press, 2000.
Reconstruction

So far sampling

Figure: From S. Mallat, A Wavelet Tour of Signal Processing

Reconstruction

Given a discrete sampled signal $f[n]$ (meaning $f(nT)$ as above), can we recover the value of $f(t)$ for any $t$?

If support of $\hat{f}(\omega)$ is in $[-\pi/T, \pi/T]$, then yes.
Reconstruction

So far sampling

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If support of $\hat{f}(\omega)$ is in $[-\pi/T, \pi/T]$, then yes.
Sampling Theorem

Shannon, Whittaker, Nyquist

If the support of $\hat{f}(\omega)$ is included in $[-\pi/T, \pi/T]$. Then

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)h_T(t - nT)$$

where $h_t(t) = \text{sinc}_T(t) = \frac{\sin(\pi/Tt)}{\pi t/T}$. 
Sampling Theorem

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To see this
Because $\hat{f}$ is band limited, $\hat{f}(\omega) = \hat{f}_T(\omega) \times 1_{[-\pi/T,\pi/T]}$. Taking the inverse FT we have that $f(t) = f_t * h_T(t)$.
Let $T = 1$:
Cons

Some reasons why sampling theorem not always used in practice:

▶ Computationally expansive: \( f(t) = \sum_{n=-\infty}^{\infty} f(nT) h_T(t-nT) \) is an infinite sum.
▶ Truncating this sum yields "ringing" artifacts.
▶ Bandlimited functions and images not always the case.
▶ Complicating issues such as digitization, noise, ...
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Interpolation

Instead of \( f(t) = \sum_{n=-\infty}^{\infty} f(nT)h_T(t - nT) \), use

\[
\tilde{f}(t) = \sum_{n=-\infty}^{\infty} c[n] \phi(t - nT)
\]

where \( \tilde{f} \) denotes an approximation of \( f(t) \).
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For interpolation require \( \tilde{f}(nT) = f(nT) \).
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For interpolation require $\tilde{f}(nT) = f(nT)$.

Main issue: how to choose coefficients $c[n]$?
Easy way:

Choose $\phi(x)$ such that $\phi(nT) = 0$ when $n \neq 0$, and $\phi(0) = 1$. 
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Generally:
In general, choose some smooth function $\phi(t)$ and set up a linear system such that

$$Bc = f$$
B-splines

B-spline degree zero
Consider the box function

\[ \beta^0(x) = 1, -\frac{1}{2} \leq x \leq \frac{1}{2} \]

with \( \beta^0(x) = 0 \) for all other \( x \).
B-splines

B-spline degree zero
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with \( \beta^0(x) = 0 \) for all other \( x \).

B-spline degree one

\[ \beta^1(x) = 1 - |x|, \quad |x| \leq 1 \]

and \( \beta^0(x) = 0 \) for all other \( x \).
B-splines degree two

\[
\beta^2(x) = \begin{cases} 
0 & \text{if } |x| > 1.5; \\
(3/2 - |x|)^2/2 & \text{if } |x| > 0.5; \\
0.75 - x^2 & \text{elsewhere.}
\end{cases}
\]
B-splines

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\end{cases} \]

B-splines defined recursively

\[ \beta^n(x) = \beta^0 \ast \beta^{n-1}(x). \]
B-splines
B-splines

Example signals

\[ \beta^0(x) \rightarrow \beta^1(x) \]
B-splines

Example signals

Important note
As $l \to \infty$, $\beta^l(x) \to \text{sinc}(x)$ in $L_p$ sense.
Solving for coefficients

As before, for finite signals, set up linear system $Bc = f$. 
Solving for coefficients

As before, for finite signals, set up linear system \( \mathbf{Bc} = \mathbf{f} \).

In 2D
As before, apply methods row by row, then column by column.

\[
f(x, y) = \sum_{m,n} c[m, n] \beta^l(x - m) \beta^l(y - n).
\]
Application: spatial transformation
Image registration

“Fuse” together intensity values from multiple images.

_Courtesy:_ Prof. Justin Crowley, Biology Dept., Carnegie Mellon University.