Introduction to Fourier Analysis

Intro. Biomedical Imaging and Image Analysis

September 3, 2008
Motivation:

Understand complex signals (functions) as a function (linear combination) of simpler ones.

\[ f(x) = \cdots + c_0 \phi_0(x) + c_1 \phi_1(x) + c_2 \phi_2(x) + \cdots \]

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- continuous signals (functions)
- discrete signals
- 1D, 2D, 3D.
For discrete signals

-Finite dimensional case: \( (N \text{ samples}) \)
  \[ s[p] = \sum_{k=0}^{N-1} c_k \phi_k[p] \]

-Infinite dimensional case (infinite sequences):
  \[ s[p] = \sum_{k=-\infty}^{\infty} c_k \phi_k[p]. \]

-For continuous signals
  \[ \sum_{k=-\infty}^{\infty} c_k \phi_k(x) \]
  \[ F(\omega) = \int_{-\infty}^{\infty} f(x) \phi_\omega(x) \]

Why are we doing this?
-Many imaging problems easier to understand if we choose building blocks (\( \phi_k \)) carefully.
-Computing the expansion coefficients (\( c_k \), or \( F(\omega) \)) above is key, but often easy.
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- Series expansion \(f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k(x)\)
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Orthonormal expansions

Basic idea

Relate the problem of finding these approximations to solving $Ax = b$.

Now $x$ expansion coefficients, $b$ is our image, and $A = [\phi_1, \phi_2, \cdots]$

When columns of $A$ are orthonormal, easy: $A^*A = I$ and therefore $x = A^Tb$.

In other notation

$b[p] = \sum_k x_k A_{p,k}$ or

$s[p] = \sum_k c_k \phi_k[p]$

c_k loosely termed Fourier coefficients.
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Discrete Fourier Transform (DFT)

Purpose
Representing finite signals of length \(N\) (\(s[n], n = 0, \ldots, N-1\)) as a linear combination of complex sinusoids.

\[
\phi_k[n] = e^{-j2\pi kn/N} = \cos(2\pi kn/N) + j\sin(2\pi kn/N)
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Fact:
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\langle \phi_k, \phi_l \rangle = \sum_{n=0}^{N-1} \phi_k^* [n] \phi_l [n] = 0 \text{ if } k \neq l \text{ and } \langle \phi_k, \phi_k \rangle = N
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In matrix notation

\[ A = [\phi_0, \phi_1, \ldots, \phi_{N-1}] \]

Therefore solving \( 1/N A c = s \) is simply \( c = A^* s \).

Use "fft", and "ifft" commands in Matlab.
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Notes About Notation

The coefficients $c[k]$ of the expansion $s[n] = \sum_{k=0}^{N-1} c[k] \varphi[k][n]$ are called the Discrete Fourier Transform of the data $s[n]$. The coefficients are often written as $\hat{s}[k]$, or $S[k]$, as opposed to $c[k]$ to indicate that these are the Fourier Transform of the signal $s[n]$. 
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All of the above for 1D finite dimensional signals.
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Perform decomposition column by column, then row by row, or vice versa.
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\[
s[m, n] = \frac{1}{M} \frac{1}{N} \sum_{k_y=0}^{M-1} \sum_{k_x=0}^{N-1} c[k_x, k_y] e^{j2\pi k_y m/M} e^{j2\pi k_x n/N}
\]
DFT example

Image

DFT

DFT example
DFT example

Explain image

As linear combination of basis vectors

The so called DFT are the coefficients of the expansion.
Convolution

Let $s$ and $h$ be two vectors of length $N$. Their convolution is defined as

$$(s * h)[n] = \sum_{k=0}^{N-1} s[k]h[n - k]$$

What happens when $n - k$ is outside of $[0, \ldots, N - 1]$?

Solution: interpret these signals as being periodic:

$$(s * h)[n] = \sum_{k=0}^{N-1} s[k]h[n - k \mod N]$$
Some Useful Properties

Convolution:

Let \( v[n] = (s * h)[n] \)

Then \( \hat{v}[k] = \hat{s}[k] \hat{h}[k] \).

Parseval's formula:

\[
\sum_{n=0}^{N-1} f[n] g[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] \hat{g}[k]
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Energy conservation: Plancherel formula:

\[
\|s\|_2^2 = \sum_{n=0}^{N-1} |s[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\hat{s}[k]|^2 = \|\hat{s}\|_2^2.
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Fourier Series

**Goal**
Decompose "nice" periodic continuous functions as sums of complex valued sinusoids.

- **Periodic functions**
  \[ f(x + T) = f(x) \]
  \( T \): period.

- **Complex sinusoids**
  - Real valued sinusoids: \( \sin(x), \cos(x) \)
  - Complex valued sinusoids: \( \cos(x) + i\sin(x) = e^{ix} \), with \( i = \sqrt{-1} \)
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Series expansion

Represent periodic continuous functions as:

\[ f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \]

with \( \omega_0 = 2\pi/T \) and \( f(t + T) = f(t) \).
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\[ a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt \]
\[ a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t) dt \]
\[ b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega_0 t) dt \]
In general:

Use

\[
\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}
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To arrive at:

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j n \omega_0 t} \]

with

\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j n \omega_0 t} dt. \]
Vector space view

Consider the vector space of square integrable functions over period $T$:

$$\int_{-T/2}^{T/2} |f(t)|^2 dt < \infty$$
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Basis

The set $\frac{1}{\sqrt{T}} e^{j\omega_0 nt}$ forms a complete orthonormal system for representing periodic functions.
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**Representation with orthonormal basis**

$$f(t) = \sum_{n=-\infty}^{\infty} \langle f(t), \phi_n(t) \rangle \phi_n^*(t)$$

with $\langle f(t), \phi_n(t) \rangle = \int_{-T/2}^{T/2} \phi_n^*(t) f(t) dt$ and $\phi_n(t) = e^{-jn\omega_0 t}$. 
Example:
Example:

![Signal Waveform](image1.png)

![Frequency Content](image2.png)
Example:
Example:
Example:
Example:
Example:

![Graph](image.png)
Example:

**Graph 1:**
- Signal
- Reconstruction

**Graph 2:**
- Signal
- Imaginary

**Frequency content**
- 200
- 100
- 0

**Graph 3:**
- Frequency
- 0
- 2
- 4
- 6
- 8
- 10
- 12
- 14
- 16
- 18
- 20
Example:
Example:

![Graphs showing Fourier analysis and frequency content](image-url)

- **Motivation**
- **Orthonormal expansions**
- **Discrete Fourier Transform**
- **Properties**
- **Fourier Series**
- **Example**
- **Convergence**
- **Discrete Time FT**
- **Summary**
Example:

![Signal and Reconstruction](image1)

![Signal and Imaginary](image2)

**Frequency content**

![Frequency content](image3)
Example:
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Example:

- **Signal**: Represents the original signal.
- **Reconstruction**: The reconstructed signal from Fourier analysis.

**Frequency Content**

- The frequency content graph illustrates the frequency components of the signal.
- The peaks correspond to the significant frequencies present in the signal.
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Biomedical Imaging and Image Analysis

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**Frequency content**
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[Graphs and diagrams showing Fourier analysis examples, including frequency content graphs and waveforms.]
Example:

- **Signal and Reconstruction:**
  - **Signal**
  - **Reconstruction**

- **Frequency Content**
  - Frequency spectrum showing peaks at certain frequencies.
Example:

![Example](image)

**Frequency content**
Example:
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Notes about convergence

What does it mean $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$?
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- Uniform convergence: Pointwise and given \( \epsilon > 0 \), there exists \( N \) s.t. \( |f^N(t) - f(t)| < \epsilon \)
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- Mean square convergence:
  $$\lim_{N \to \infty} \sqrt{\int_{-T/2}^{T/2} |f(t) - f^N(t)|^2} = 0$$
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\lim_{N \to \infty} \sqrt{\int_{-T/2}^{T/2} |f(t) - f^N(t)|^2} = 0
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Fourier Series
Notes about convergence

What does it mean \( f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \)?

Let \( f^N(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} \):

- **Pointwise convergence**: \( \lim_{N \to \infty} f^N(t) = f(t) \).
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- If \( f(t) \) continuous: series converges uniformly.
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Fourier Series

- If \( f(t) \) continuous: series converges uniformly.
- If \( f(t) \) discontinuous: mean square convergence.
Discrete Time Fourier Transform

Consider the vector space of infinite square summable sequences $s[n]$, $n = \cdots, -1, 0, 1, 2, \cdots$. 
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**DTFT**

\[
\hat{s}(\omega) = \sum_{n=-\infty}^{\infty} s[n] e^{-jn\omega}
\]

**Inverse**

\[
s[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{s}(\omega) e^{j\omega n} d\omega
\]
Properties

Convolution

Let $c[n] = \sum_{k=-\infty}^{\infty} s[k] h[n-k]$. Then $\hat{c}[k] = \hat{s}[k] \hat{h}[k]$.

Parseval's formula:

$\sum_{n=-\infty}^{\infty} f[n] g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{g}(\omega) d\omega$

Energy conservation: Plancherel formula:

$\|s\|_2^2 = \sum_{n=-\infty}^{\infty} |s[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{s}(\omega)|^2 d\omega$
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In discrete time case (infinite sequences) the Fourier Transform is interpreted as a frequency response. That is, how much of a contribution from a particular sequence \( e^{j\omega k} \) there is in the time series: \( \omega \) is the frequency.
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**Notation**

Sometimes the DTFT of sequence $s$ is denoted $S(e^{j\omega})$ due to connections to the Z-transform.

**N-D**

Extend to 2D and 3D, and beyond, by applying the operations sequentially over each dimension.
Fourier Transform

Goal
Represent a "nice" continuous function $f(t)$ with complex sinusoids $e^{j\omega t}$.

\[
 f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{j\omega t} d\omega.
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\[
 \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \langle f, e^{j\omega t} \rangle
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Convergence issues
- If $f(t)$ continuous, formula above is exact.
- If not, the formulas hold in the least squares sense.
Properties

Convolution

- \( g = h \ast f = \int_{-\infty}^{\infty} f(t - u)h(u)du \)
- Then \( \hat{g}(\omega) = \hat{h}(\omega)\hat{f}(\omega) \)
Properties

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Parseval

\[
\langle f, h \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{h} \rangle.
\]

Also implies Plancherel.
In two dimensions

As before, perform it along each dimension individually.

\[
\hat{f}(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{j(\omega_x x + \omega_y y)} \, dx \, dy.
\]

\[
f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega_x, \omega_y) e^{j(\omega_x x + \omega_y y)} \, d\omega_x \, d\omega_y.
\]
Properties

**Table: Fourier transform properties**

<table>
<thead>
<tr>
<th>Property</th>
<th>Function</th>
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<tr>
<td>Multiplication</td>
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<td>$\frac{1}{2\pi} \hat{f}_1(\omega) \hat{f}_2(\omega)$</td>
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<td>Translation</td>
<td>$f(t - t_0)$</td>
<td>$e^{-jt_0\omega} \hat{f}(\omega)$</td>
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<tr>
<td>Modulation</td>
<td>$e^{j\omega_0 t} f(t)$</td>
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<td>Scaling</td>
<td>$f\left(\frac{t}{s}\right)$</td>
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<tr>
<td>Time derivatives</td>
<td>$f^p$</td>
<td>$(j\omega)^p \hat{f}(\omega)$</td>
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<tr>
<td>Frequency derivatives</td>
<td>$(-jt)^p f(t)$</td>
<td>$\hat{f}^{(p)}(\omega)$</td>
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Note: most of these can be proved with a change of variable in the Fourier integral.
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One can use complex sinusoids $e^{j\omega n}$ or $e^{k\omega t}$ to represent, explain signals.
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- Fourier Series: periodic functions.
- Fourier Transform (FT). Functions.
Summary

These have important properties.

- Convolution (filtering).
- Parseval (inner product).
- Plancherel (energy).
- Linearity.
- Many others ...

We will see many applications of these concepts, including in image acquisition, as well as processing.
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